UT-HEP-696-94
hepth/9412226

# Calculation of Gromov-Witten invariants for $CP^3, CP^4$,and $Gr(2,4)$

Masao Jinzenji and Yi Sun

*Department of Phisics, University of Tokyo*

*Bunkyo-ku, Tokyo 113, Japan*

**Abstract**

Using the associativity relations of the topological Sigma Models with target spaces, $CP^3, CP^4$ and $Gr(2,4)$ , we derive recursion relations of their correlation and evaluate them up to certain order in the expansion over the instantons. The expansion coeffieients are regarded as the number of rational curves in $CP^3, CP^4$ and $Gr(2,4)$ which intersect various types of submanifolds corresponding to the choice of BRST invariant operators in the correlation functions.

# 1 Introduction

Recently various methods for calculating correlation functions of topological Sigma Models (A-Model) have been obtained from the analysis of their structure as the twisted version of $N = 2$ super conformal field theory. One of the most striking feature is Mirror Symmetry,which emerged from the analysis of spectrum of 2-types of rings (chiral,chiral) and (chiral,antichiral) of Topological Sigma Model on Calabi-Yau mfd and from the speculation that there are pairs of Calabi-Yau manifold $M$ and $\tilde{M}$ such that (chiral,chiral) ring of $M$ (resp.$\tilde{M}$) and (chiral,antichiral) ring of $\tilde{M}$ (resp.$M$) coincide. Another feature is recently found by Kontsevich, Manin [1] and Dubrovin [5] ,which uses the associativity of chiral algebra for Topological Sigma Models (coupled with gravity but on small phase space). In [1], the authors defined Gromov-Witten classes as cohomology classes of moduli space of punctured $CP^1$ and showed that associativity condition is reduced to the characteristic of boundaries of the moduli space under the assumption of splitting axiom. Using the method of algebraic geometry numbers of rational curves in M are counted which pass through Poincare duals of cohomology classes corresponding to inserted chiral primary fields. They showed that for Fano varieties $M$ whose cohomology are generated by $H^2(M)$, the associativity condition is enough for the construction of all correlation functions.

With these results available we explicitcy calculate correlation functions of $CP^3$,$CP^4$ and $Gr(2, 4)$. In the case of $CP^3$ and $CP^4$ we obtain a series of overdetermined differential equation for the free energy. We assume the expansion of the free energy as a generating function of topological amplitudes and impose ghost number selection rule. Then we pick a subset of these equations with linear terms and derive recurtion relations which determine the correlation functions coming from moduli space of maps of degree $d$ from the ones of lower degree. Then we observe that all of them are determined and calculate them to several degree d. For $CP^3$,$d = 6$,$CP^4$,$d = 4$,and for $Gr(2, 4)$,$d = 4$. We checked compatibility of all the equations in our calculation up to the degree we calculated.

In section 1, we introduce the interpretation of correlation functions as intersection numbers of holomorphic maps from $CP^1$ to the target space $M$ originally derived by Witten [8] and explain when coupled to gravity, they count the number of rational curves passing through Poincaré duals of inserted operators. In section 2, assuming that these models are twisted version of $N = 2$ super conformal field theory and have stable property of chiral algebra under pertubation of chiral primaary fields, we set up the calculation of the free energy $F_M$ using associativity of this algebra. In section 3, we carry out the calculation for each case $CP^3$,$CP^4$ and $Gr(2, 4)$. Recurtion relations which are used in determination of amplitudes are written out explicitly. All results are collected in tables at the end of this paper.

## 2 Meaning of the Correlation Function

In the topological sigma model (A-model) which describes maps from $CP^1$ to the target space $M$, BRST-closed observables are constructed from elements of $H^*(M)$ [7]. We denote the BRST-closed observable constructed from $W \in H^*(M)$ as $\mathcal{O}_W$. Witten showed in the pure matter case [8] (without coupling to gravity) topological correlation functions are given in terms of intersection numbers of holomorphic maps from $CP^1$ to $M$ as follows.

$$\begin{aligned}
\langle \mathcal{O}_{W_{i_1}}(z_1)\mathcal{O}_{W_{i_2}}(z_2)\cdots\mathcal{O}_{W_{i_k}}(z_k)\rangle &= \int_{\mathcal{M}_d(M)} \chi(\nu)\prod_{j=1}^{k}\phi_j^*(W_{i_j}) \qquad (2.1)\\
\phi_j : \mathcal{M}_d(M) &\mapsto M \\
f \in \mathcal{M}_d(M), z_j &\mapsto f(z_j) \in M \quad j = 1,\cdots,k
\end{aligned}$$

($\mathcal{M}_d(M)$ is the moduli space of holomorphic maps from $CP^1$ to $M$ of degree d,and $(z_1,\ldots,z_k)$ are "fixed" distinct points on $CP^1$. Degree $d$ is related to the sum of $dim_C(W_{i_j})$ by the topological selection rule which we will introduce later ).

$\nu$ is the additional degree of freedom which arises when f can be decomposed as $f = \tilde{f}\circ\alpha$ where $\alpha$ is a map from $CP^1$ to $CP^1$ of degree $\frac{d}{j}$ and $\tilde{f}$ a map from $CP^1$ to $M$ of degree $j(d|j)$. But as we will discuss later, we have to consider $\nu$ only when $M$ is C.Y manifold, i.e. $c_1(TM) = 0$.

Since $\phi_j^*(W_{i_j})$ defines $dim_C(W_{i_j})$ form on $\mathcal{M}_d(M)$, in generic case when $\nu$ is trivial, $\langle\mathcal{O}_{W_{i_1}}\cdots\mathcal{O}_{W_{i_k}}\rangle$ doesn't vanish only when the following conditions are satisfied.

$$\begin{aligned}
\sum_{j=1}^{k} dim_C(W_{i_j}) &= dim_C(\mathcal{M}_d(M)) \\
&= dimH^0(f^*(TM)) \\
&= dc_1(TM) + dim_C(M) \qquad (2.2)
\end{aligned}$$

In deriving the third line from the second line, we used Rieman-Roch theorem and assumed $H^1(f^*(TM)) = 0$.

If we take $W_{i_j}$ as the form which has a delta-function support on the Poincare dual of $W_{i_j}$, $PD(W_{i_j})$, we can interpret $\phi_i^*(W_{i_j})$ as the following constraint on $\mathcal{M}_d(M)$.

$$f(z_j) \in PD(W_{i_j}) \qquad (2.3)$$

We can easily see the above condition imposes $dim_C(W_i)$ independent constraints on $\mathcal{M}_d(M)$ (use count degrees of freedom in the complex sense). Since we have to use $(dim_C(M) - dim_C(f(CP^1)) - dim_C(PD(W_{i_j}))$ degrees of freedom to let $f(CP^1)\cap PD(W_{i_j}) \neq \emptyset$ and in case $dim_C(f(CP^1)) = 1$, we have to use one further degree of freedom to let $z_j$ to lie on $f(CP^1)\cap PD(W_{i_j})$. Condition (2.3) tells us that by imposing all the constraints $i = 1,\cdots,k$, we have zero degrees of freedom and topological

correlation functions reduce to

$$\begin{aligned}&\langle \mathcal{O}_{W_{i_1}}(z_1)\cdots\mathcal{O}_{W_{i_k}}(z_k)\rangle_{generic}\\ &\quad = {}^\sharp\{f : CP^1 \xrightarrow{hol.} M | f(z_j) \in PD(W_{i_j}), j = 1,\cdots,k\}\end{aligned} \tag{2.4}$$

At this point, we consider the case of multiple cover map. From the above argument, multiple cover map $f = \tilde{f}\circ\alpha$ also has to satisfy the condition (2.3) which restricts the motion of $f(CP^1) = \tilde{f}(CP^1)$ in the target space M. But since $\tilde{f}$ is a map of degree j, it has as many as $jc_1(TM) + dim_C(M) (< dc_1(TM) + dim_C(M))$ freedom in M and this is imcompatible with (2.3). Only when $c_1(TM) = 0$ i.e. $M$ is C.Y manifold, compatibility of (2.2) and (2.3) holds in the case of multiple cover map and we have to integrate the additional $\nu$. Then we conclude that when $c_1(TM) > 0$,we can neglect $\chi(\nu)$ and only consider the generic case.

Next, let us consider what happens if we couple topological gravity to the above topological sigma model. Roughly speaking, we have to integrate over moduli space of $CP^1$ with punctures. Since the moduli space of $CP^1$ with $k$-punctures are given by the position of $k$-distinct points on $CP^1$ divided by $SL(2, C)$, which is the internal symmetry group of $CP^1$, the difinition (2.2) is modified as follows.

$$\begin{aligned}\langle \mathcal{O}_{W_{i_1}}\cdots\mathcal{O}_{W_{i_k}}\rangle &= \int_{\mathcal{M}_{d,0,k}(M)}\prod_{j=1}^{k}\tilde{\phi}_j^{\,*}(W_{i_j}) \qquad (2.5)\\ \tilde{\phi}_j : \mathcal{M}_{d,0,k}(M) &\mapsto M\\ (z_1, z_2,\cdots, z_k, f)/SL(2, C) &\mapsto (f(z_1),\cdots, f(z_k))\end{aligned}$$

where the action of $u \in SL(2, C)$ is defined as follows.

$$u\circ(z_1, z_2,\cdots, z_k, f) = (u(z_1), u(z_2),\cdots, u(z_k), (u^{-1})^* f) \tag{2.6}$$

This action is natural in the sense that the image of $\tilde{\phi}_j$ remains invariant under $SL(2, C)$. Main difference between (2.4) and (2.6) is that in the former case, we keep $z_i$ "fixed" on $CP^1$ but in the latter they move. Then we have $dim_C(\mathcal{M}_{d,0,k}(M)) = k - 3 + dc_1(TM) + dim_C(M)$ and modify (2.2) as follows.

$$\begin{aligned}&\sum_{j=1}^{k} dim_C(W_{i_j}) = dc_1(TM) + dim_C(M) + k - 3\\ \Longleftrightarrow &\sum_{j=1}^{k} dim_C(W_{i_j} - 1) = dc_1(TM) + dim_C(M) - 3\end{aligned} \tag{2.7}$$

Integrating over the positions of $k$-punctures the codition (2.3) changes into

$$f(CP^1)\cap PD(W_{i_j}) \neq \emptyset. \tag{2.8}$$

Under the condition (2.7), $f(CP^1)\cap PD(W_{i_j})$ must be a finite point set for each $j$ and $z_i$ integration contributes $(f(CP^1)\cap PD(W_{i_j}))^\sharp$ to the correlation function. Then

we have

$$\langle \mathcal{O}_{W_{i_1}} \cdots \mathcal{O}_{W_{i_k}} \rangle = \sum_f \prod_{j=1}^{k} (f(CP^1) \cap PD(W_{i_j}))^{\sharp}$$

$$\{f : CP^1 \xrightarrow{hol.} M | f(CP^1) \cap PD(W_{i_j}) \neq \emptyset \quad j = 1, 2, \cdots, k\} \tag{2.9}$$

# 3 Set up of the calculation

Topological Sigma Model (A-Model) can be constructed as the twisted version of $N = 2$ super conformal field theory [9]. We can perturb topological field theory by adding the terms $\sum_i t_i \mathcal{O}_{W_i}$ to the lagrangian and correlation functions depend on variables $\{t_i\}$ [4].

$$\begin{aligned} &\langle \mathcal{O}_{W_{i_1}} \mathcal{O}_{W_{i_2}} \cdots \mathcal{O}_{W_{i_k}} (t_1, t_2, \ldots, t_{dim(H^*(M))}) \rangle \\ &= \int \mathcal{D}X e^{-L(X) + \sum_i t_i \mathcal{O}_{W_i}} \mathcal{O}_{W_{i_1}} \mathcal{O}_{W_{i_2}} \cdots \mathcal{O}_{W_{i_k}} \end{aligned} \tag{3.1}$$

where X denotes the field variables of the A-Model. We set $D = dim(H^*(M))$. As the twisted version of N=2 SCFT (coupled with gravity and on small phase space ),$\{\mathcal{O}_{W_i}\}$ has ring structure which can be determined from three point correlation functions.

$$\mathcal{O}_{W_i} \mathcal{O}_{W_j} = C_{ij}^k (t_1, t_2, \ldots, t_D) \mathcal{O}_{W_k} \tag{3.2}$$

$$\text{where} \quad C_{ij}^k = C_{ijl} \eta^{lk} \tag{3.3}$$

$$C_{ijl} = \langle \mathcal{O}_{W_i} \mathcal{O}_{W_j} \mathcal{O}_{W_l} (t_1, t_2, \ldots, t_D) \rangle \tag{3.4}$$

$$\eta^{kl} \eta_{lm} = \delta_m^k \tag{3.5}$$

$$\eta_{lm} = C_{1lm} (t_1, t_2, \ldots, t_D) \tag{3.6}$$

In our notation $W_1$ corresponds to $1 \in H^*(M)$ and we set $W_2$ to the Kähler form of $M$ (in our case where $M$ is $CP^3, CP^4$ or $Gr(2,4)$, $dim(H^2(M)) = dim(H^{1,1}(M)) = 1$,this notation is O.K). We assume that $t_i$'s are flat coordinates and $\eta_{lm}$ do not depend on them and determined by classical intersection number $\int_M W_l \wedge W_m$. Next, we impose associativity condition on this algebra.(This relation is DWVV eq.)

$$\begin{aligned} & (\mathcal{O}_{W_i} \mathcal{O}_{W_j}) \mathcal{O}_{W_k} = \mathcal{O}_{W_i} (\mathcal{O}_{W_j} \mathcal{O}_{W_k}) \\ \Longleftrightarrow \quad & C_{ij}^l \mathcal{O}_{W_l} \mathcal{O}_{W_k} = \mathcal{O}_{W_i} C_{jk}^m \mathcal{O}_{W_m} \\ \Longleftrightarrow \quad & C_{ij}^l C_{lk}^n \mathcal{O}_{W_n} = C_{jk}^m C_{im}^n \mathcal{O}_{W_n} \\ \Longleftrightarrow \quad & C_{ij}^l C_{lk}^n = C_{jk}^m C_{im}^n \\ \Longleftrightarrow \quad & C_{ijm} \eta^{lm} C_{lkn} = C_{jkm} \eta^{lm} C_{imn} \end{aligned} \tag{3.7}$$

And there exists a free energy (prepotential) $F_M(t_1, \ldots, t_D)$ which satisfies following conditions.

$$C_{ijk}(t_1, t_2, \ldots, t_D) = \partial_i \partial_j \partial_k F_M \tag{3.8}$$

$$(\partial_i := \frac{\partial}{\partial t_i}) \tag{3.9}$$

Combining (3.7) and (3.9), we obtain a series of partial differential equations for $F_M$.

$$\eta^{lm}\partial_i\partial_j\partial_m F_M\partial_l\partial_k\partial_n F_M = \eta^{lm}\partial_j\partial_k\partial_m F_M\partial_i\partial_l\partial_n F_M \tag{3.10}$$

We can also consider prepotential as the generating function of all topological correlation functions.

$$F_M(t_1,\ldots,t_D) = \sum_{n_1,\cdots,n_D\geq 0}\langle\mathcal{O}_{W_1}^{n_1}\cdots\mathcal{O}_{W_D}^{n_D}\rangle\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_D^{n_D}}{n_D!} \tag{3.11}$$

($\mathcal{O}_{W_i}^{n_i}$ represents $\overbrace{\mathcal{O}_{W_i}\cdots\mathcal{O}_{W_i}}^{n_i \text{ times}}$ and should not be confused with $\mathcal{O}_{(W_i)^{n_i}}$ ). At the topological point (i.e., all the $t_i$'s are set to zero) correlation functions become intersection numbers on moduli spaces of holomorphic maps from $CP^1$ (with $k$– marked points) to target space $M$.

Holomorphic maps $f$ are characterized by their degree which equal the intersection number of $f^*(CP^1)$ with the Kähler form of target space $M$. Then $\langle\mathcal{O}_{W_1}^{n_1}\cdots\mathcal{O}_{W_D}^{n_D}\rangle$ remains non–zero only when the following topological selection rule is satisfied.

$$\begin{aligned}\sum_{i=1}^{D} n_i dim_C(W_i) &= \sum_{i=1}^{D} n_i - 3 + dimH^0(CP^1,\phi^*(TM))\\ \Longleftrightarrow \sum_{i=1}^{D} n_i(dim_C(W_i)-1) &= -3 + dc_1(TM) + dim_C(M)\end{aligned} \tag{3.12}$$

Here d is the degree of holomorphic map and we used Riemann–Roch theorem in deriving the second line from the first one. When d equals zero, $f$ is a constant map to the target space and moduli space becomes just the direct product of target space and moduli space of $CP^1$ with $\sum_{i=1}^{D} n_i$ punctures. Then selection rules (3.12) decomposes to

$$\sum_{i=1}^{D} n_i dim_C(W_i) = dim_C(M) \tag{3.13}$$

and

$$\sum_{i=1}^{D} n_i = 3 \tag{3.14}$$

From (3.14), we conclude that in $d=0$ case only 3-point functions survive and correlation functions are just classical intersection numbers $\int_M W_i\wedge W_j\wedge W_k$.

From the flat metric condition, insertion of $W_1$ remains non-zero only for three point functions from constant maps because one and two point functions including $W_1$ cannot remein nonzero when $c_1(M)\geq 1$ and $d\geq 1$ and if we suppose $n(\geq 3)$ point functions remain nonzero in $d\geq 1$ sector, flat metric condition is broken(in three point functions in $d\geq 1$ sector, we take into account of the insertion of operator $\mathcal{O}_{W_2}(dim_C(W_2)=1)$ which we will discuss later).

With these considerations, expansion of the free energy becomes

$$F_M(t_1,\ldots,t_D) = \frac{1}{6}\int_M (\sum_{i=1}^{D} t_i W_i)^3 + \sum_{d=1}^{\infty}\sum_{n_2,\cdots,n_D\geq 0}\frac{{t_2}^{n_2}}{n_2!}\cdots\frac{{t_D}^{n_D}}{n_D!}\langle\mathcal{O}_{W_2}^{n_2}\cdots\mathcal{O}_{W_D}^{n_D}\rangle$$
$$(\sum_{i=2}^{D} n_i(dim_C(W_i)-1) = -3 + dc_1(TM) + dim_C(M)) \quad (3.15)$$

(where the product of the first term of r.h.s. means taking the wedge product of $H^*(M)$).

Next, we consider the insertion of the operator $\mathcal{O}_{W_2}$ which corresponds to the Kähler form of the target space . Since $codim_C(PD(W_2)) = 1$, holomorphic map $f$ of degree $d$ always intersects with it in $d$-points,and the condition $f(CP^1)\cap PD(W_2)\neq\emptyset$ imposes no constraint. Then from (2.9) the insertion of $\mathcal{O}_{W_2}$ results in the multiplication by a factor d,

$$\langle\mathcal{O}_{W_2}^{n_2}\mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle = d^{n_2}\langle\mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle \quad (3.16)$$

Combining (3.15) and (3.16) we obtain the following expansion

$$F_M(t_1,\ldots,t_D) = \frac{1}{6}\int_M (\sum_{i=1}^{D} t_i W_i)^3 + \sum_{d=1}^{\infty}\sum_{n_3,\cdots,n_D\geq 0}\frac{{t_3}^{n_3}}{n_3!}\cdots\frac{{t_D}^{n_D}}{n_D!}\langle\mathcal{O}_{W_3}^{n_3}\cdots\mathcal{O}_{W_D}^{n_D}\rangle e^{dt_2}$$
$$(\sum_{i=3}^{D} n_i(dim_C(W_i)-1) = -3 + dc_1(TM) + dim_C(M)) \quad (3.17)$$

Then by combining (3.10) and (3.17), we determine the correlation functions in the case where target space is $CP^3, CP^4$ and $Gr(2,4)$.

# 4 The Calculations

The non zero Betti numbers of $CP^3$, $CP^4$ and $Gr(2,4)$ are

$$b_{00} = b_{11} = b_{22} = b_{33} = 1 \quad (4.1)$$

$$b_{00} = b_{11} = b_{22} = b_{33} = b_{44} = 1 \quad (4.2)$$

$$b_{00} = b_{11} = b_{33} = b_{44} = 1, \qquad b_{22} = 2 \quad (4.3)$$

respectively. By means of wedge product we obtain an associative, commutative ring $H^*(M,Q)$ for each manifold $M$.For $CP^3$, $CP^4$, and $Gr(2,4)$ we have the multiplication table as follows.

Table 1: **The ring of $CP^3$**

| | $W_1$ | $W_2$ | $W_3$ | $W_4$ |
|---|---|---|---|---|
| $W_1$ | $W_1$ | $W_2$ | $W_3$ | $W_4$ |
| $W_2$ | $W_2$ | $W_3$ | $W_4$ | 0 |
| $W_3$ | $W_3$ | $W_4$ | 0 | 0 |
| $W_4$ | $W_4$ | 0 | 0 | 0 |

$dim_{\mathcal{C}}(W_1)$=0 $dim_{\mathcal{C}}(W_2)$=1

$dim_{\mathcal{C}}(W_3)$=2 $dim_{\mathcal{C}}(W_4)$=3

Table 2: **The ring of $CP^4$**

| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ |
|---|---|---|---|---|---|
| $W_1$ | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ |
| $W_2$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | 0 |
| $W_3$ | $W_3$ | $W_4$ | $W_5$ | 0 | 0 |
| $W_4$ | $W_4$ | $W_5$ | 0 | 0 | 0 |
| $W_5$ | $W_5$ | 0 | 0 | 0 | 0 |

$dim_{\mathcal{C}}(W_1)$=0 $dim_{\mathcal{C}}(W_2)$=1

$dim_{\mathcal{C}}(W_3)$=2 $dim_{\mathcal{C}}(W_4)$=3 $dim_{\mathcal{C}}(W_5)$=4

Dual to each cohomology class is a class of cycles (e.g. for the case of $CP^3$ $W_3$ is dual to a point, $W_2$ is dual to a line, $W_1$ is dual to the $CP^3$).

As a point intersects the $CP^3$ in a point and a line intersects the plane by a point. Thus we have for $CP^3$,

$$< W_1, W_4 >= 1, < W_2, W_3 >= 1 \tag{4.4}$$

For $CP^4$

$$< W_1, W_5 >= 1, \qquad < W_2, W_4 >= 1, \qquad < W_3, W_3 >= 1 \tag{4.5}$$

and for Gr(2,4) it becomes

$$< W_1, W_6 >= 1, \qquad < W_2, W_5 >= 1 \tag{4.6}$$

$$< W_3, W_3 >= 1, \qquad < W_4, W_4 >= 1 \tag{4.7}$$

All other intersections on generators being zero. The $CP^3$, $CP^4$ ring can be identified the ring of polynomials in one indeterminate C[x] modulo the gradient of

$$W(x) = x^4/4, \qquad W(x) = x^5/5 \tag{4.8}$$

Table 3: **The ring of** $Gr(2,4)$

| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ |
|---|---|---|---|---|---|---|
| $W_1$ | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ |
| $W_2$ | $W_2$ | $W_3+W_4$ | $W_5$ | $W_5$ | $W_6$ | 0 |
| $W_3$ | $W_3$ | $W_5$ | $W_6$ | 0 | 0 | 0 |
| $W_4$ | $W_4$ | $W_5$ | 0 | 0 | 0 | 0 |
| $W_5$ | $W_5$ | $W_6$ | 0 | 0 | 0 | 0 |
| $W_6$ | $W_6$ | 0 | 0 | 0 | 0 | 0 |

$dim_{\mathcal{C}}(W_1)$=0 $dim_{\mathcal{C}}(W_2)$=1 $dim_{\mathcal{C}}(W_3)$=2

$dim_{\mathcal{C}}(W_4)$=2 $dim_{\mathcal{C}}(W_5)$=3 $dim_{\mathcal{C}}W_6$=4

Ine the case of Grassmanians their cohomology $H^*(Gr,Q)$ can't be generated by $H^2(Gr,Q)$. The cohomology ring of Grassmanian $Gr(2,4)$ for instance, can be written as the singularity ring generated by a single potential[6]

$$W(x_i)=\frac{1}{5}x_1^5-x_1^3x_2+x_2^2x_1 \tag{4.9}$$

Where $x_1$ correspond to $W_2$ and $x_2$ correspond to $\frac{1}{2}(W_3+W_4)$.

From (3.6) one can split $F_M$ into a classical part and instanton correction part as

$$F_M=f_{cl}+f_M \tag{4.10}$$

So for $CP^3$, $CP^4$ and $Gr(2,4)$ we have

$$F_{CP^3}=\frac{1}{2}t_1^2t_4+t_1t_2t_3+\frac{1}{6}t_2^3+f_{CP^3}(t_2,t_3,t_4) \tag{4.11}$$

$$F_{CP^4}=\frac{1}{2}t_1^2t_5+\frac{1}{2}t_1t_4^2+\frac{1}{2}t_3^2t_4+t_1t_2t_5+f_{CP^4}(t_2,t_3,t_4,t_5) \tag{4.12}$$

$$\begin{aligned}F_{Gr(2,4)}=&\frac{1}{2}t_1^2t_6+\frac{1}{2}t_1t_3^2+\frac{1}{2}t_1t_4^2+t_1t_1t_5+\frac{1}{2}t_2^2t_3+\frac{1}{2}t_2^2t_4\\&+f_{Gr(2,4)}(t_2,t_3,t_4,t_5,t_6)\end{aligned} \tag{4.13}$$

From (2.7) the Riemann-Roch theorem tell us

$$dimH^0(CP^1,f^*(TM))-3=(dimM-3)+dc_1(TM) \tag{4.14}$$

Once we specify the target space , we know its frist Chern class, then the above formula give the dimension of its moduli space. In case of $CP^3$ $c_1(TCP^3)=4$ so,

$$dimH^0(CP^1,f^*(TCP^3))-3=4d \tag{4.15}$$

For $CP^4$ and $Gr(2,4)$, the frist Chern class are

$$c_1(CP^4) = 5, \qquad c_1(TGr(2,4)) = 4 \tag{4.16}$$

Thus

$$dimH^0(CP^1, f^*(TCP^4)) - 3 = 5d + 1 \tag{4.17}$$

$$dimH^0(CP^1, f^*(TGr(2,4))) - 3 = 4d + 1 \tag{4.18}$$

From (3.17) we can expond $f_M$ further as follows

$$\begin{aligned} f_{CP^3} &= \sum_{d=1}^{\infty} \sum_{n_3+2n_4=4d} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} >}{n_3! n_4!} t_3^{n_3} t_4^{n_4} e^{dt_2} \\ &= \sum_{d=1}^{\infty} \sum_{n_4} \frac{< \mathcal{O}_{W_3}^{4d-2n_4} \mathcal{O}_{W_4}^{n_4} >}{(4d-2n_4)! n_4!} t_3^{4d-2n_4} t_4^{n_4} e^{dt_2} \end{aligned} \tag{4.19}$$

$$\begin{aligned} f_{CP^4} &= \sum_{d=1}^{\infty} \sum_{n_3+2n_4+3n_5=5d+1} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >}{n_3! n_4! n_5!} t_3^{n_3} t_4^{n_4} t_5^{n_5} e^{dt_2} \\ &= \sum_{d=1}^{\infty} \sum_{n_4,n_5} \frac{< \mathcal{O}_{W_3}^{5d-2n_4-3n_5+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >}{(5d-2n_4-3n_5+1)! n_4! n_5!} t_3^{5d-2n_4-3n_5+1} t_4^{n_4} t_5^{n_5} e^{dt_2} \end{aligned} \tag{4.20}$$

$$\begin{aligned} f_{Gr(2,4)} &= \sum_{d=1}^{\infty} \sum_{n_3+n_4+2n_5+3n_6=4d+1} \frac{< \mathcal{O}_{W_3}^{n_3} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >}{n_3! n_4! n_5! n_6!} t_3^{n_3} t_4^{n_4} t_5^{n_5} t_6^{n_6} e^{dt_2} \\ &= \sum_{d=1}^{\infty} \sum_{n_4,n_5,n_6} \frac{< \mathcal{O}_{W_3}^{4d-n_4-2n_5-3n_6+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >}{(4d-n_4-2n_5-3n_6+1)! n_4! n_5!} t_3^{4d-n_4-2n_5-3n_6+1} t_4^{n_4} t_5^{n_5} t_6^{n_6} e^{dt_2} \end{aligned} \tag{4.21}$$

Then, we abbreviate the notion in the following calculation as

$$< \mathcal{O}_{W_3}^{4d-2n_4} \mathcal{O}_{W_4}^{n_4} >_{CP^3} = N_{n_4}^d \tag{4.22}$$

$$< \mathcal{O}_{W_3}^{5d-2n_4-3n_5+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} >_{CP^4} = N_{n_4,n_5}^d \tag{4.23}$$

$$< \mathcal{O}_{W_3}^{4d-n_4-2n_5-3n_6+1} \mathcal{O}_{W_4}^{n_4} \mathcal{O}_{W_5}^{n_5} \mathcal{O}_{W_6}^{n_6} >_{Gr(2,4)} = N_{n_4,n_5,n_6}^d \tag{4.24}$$

We let $t_2 = x, t_3 = y, t_4 = z$ for $CP^3$,$t_2 = w, t_3 = x, t_4 = y, t_5 = z$ for $CP^4$ and $t_2 = v, t_3 = w, t_4 = x, t_5 = y, t_6 = z$ for $Gr(2,4)$. A deformation of the multiplication table (table 1, table 2, table 3) become the fusion rules for the quantum cohomology ring with $\mathcal{O}_{W_i}$'s substituted for $t$'s as

$$\mathcal{O}_{W_i} \circ \mathcal{O}_{W_j} = \partial_i \partial_j \partial_l F_M \eta^{lk} \mathcal{O}_{W_k} \tag{4.25}$$

The structure constants of the quantum cohomology obey the so called WDVV equation which satisfying the requirements[3]

(i)commutativity

(ii)associativity

(iii)existence of a unit $\mathcal{O}_{W_1}$

Commutativity follows from the definition, while condition(3.6) (equivalently(3.17) expresses that $\mathcal{O}_{W_1}$ plays the role of unit. The crucial assumption is the associativity which imposes strong conditions on $f_M$. Now let us introduce some more notations, by $f_{M,xyz}$ we mean $\partial_x\partial_y\partial_z f_M$. In the following we will simply omit the index "M", and just denote it as $f_{xyz}$.

The quantum ring of $CP^3$ is

$$\mathcal{O}_{W_2}\mathcal{O}_{W_2} = f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + (f_{xxx}+1)\mathcal{O}_{W_3}, \tag{4.26a}$$

$$\mathcal{O}_{W_2}\mathcal{O}_{W_3} = f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + (f_{xxy})\mathcal{O}_{W_3} + \mathcal{O}_{W_4}, \tag{4.26b}$$

$$\mathcal{O}_{W_2}\mathcal{O}_{W_4} = f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + (f_{xxz})\mathcal{O}_{W_3}, \tag{4.26c}$$

$$\mathcal{O}_{W_3}\mathcal{O}_{W_3} = f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + (f_{xyy})\mathcal{O}_{W_3}, \tag{4.26d}$$

$$\mathcal{O}_{W_3}\mathcal{O}_{W_4} = f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + (f_{xyz})\mathcal{O}_{W_3}, \tag{4.26e}$$

$$\mathcal{O}_{W_4}\mathcal{O}_{W_4} = f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + (f_{xzz})\mathcal{O}_{W_3}. \tag{4.26f}$$

The quantum ring of $CP^4$ is

$$\mathcal{O}_{W_2}\mathcal{O}_{W_2} = f_{wwz}\mathcal{O}_{W_1} + f_{wwy}\mathcal{O}_{W_2} + f_{wwx}\mathcal{O}_{W_3} + (f_{www}+1)\mathcal{O}_{W_4}, \tag{4.27a}$$

$$\mathcal{O}_{W_2}\mathcal{O}_{W_3} = f_{wxz}\mathcal{O}_{W_1} + f_{wxy}\mathcal{O}_{W_2} + f_{wxx}\mathcal{O}_{W_3} + f_{wwx}\mathcal{O}_{W_4}, \tag{4.27b}$$

$$\mathcal{O}_{W_2}\mathcal{O}_{W_4} = f_{wyz}\mathcal{O}_{W_1} + f_{wyy}\mathcal{O}_{W_2} + f_{wxy}\mathcal{O}_{W_3} + f_{wwy}\mathcal{O}_{W_4} + \mathcal{O}_{W_5}, \tag{4.27c}$$

$$\mathcal{O}_{W_2}\mathcal{O}_{W_5} = f_{wzz}\mathcal{O}_{W_1} + f_{wyz}\mathcal{O}_{W_2} + f_{wxz}\mathcal{O}_{W_3} + f_{wwz}\mathcal{O}_{W_4}, \tag{4.27d}$$

$$\mathcal{O}_{W_3}\mathcal{O}_{W_3} = f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + f_{xxx}\mathcal{O}_{W_3} + f_{wxx}\mathcal{O}_{W_4} + \mathcal{O}_{W_5}, \tag{4.27e}$$

$$\mathcal{O}_{W_3}\mathcal{O}_{W_4} = f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + f_{xxy}\mathcal{O}_{W_3} + f_{wxy}\mathcal{O}_{W_4}, \tag{4.27f}$$

$$\mathcal{O}_{W_3}\mathcal{O}_{W_5} = f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + f_{xxz}\mathcal{O}_{W_3} + f_{wxz}\mathcal{O}_{W_4}, \tag{4.27g}$$

$$\mathcal{O}_{W_4}\mathcal{O}_{W_4} = f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + f_{xyy}\mathcal{O}_{W_3} + f_{wyy}\mathcal{O}_{W_4}, \tag{4.27h}$$

$$\mathcal{O}_{W_4}\mathcal{O}_{W_5} = f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + f_{xyz}\mathcal{O}_{W_3} + f_{wyz}\mathcal{O}_{W_4}, \tag{4.27i}$$

$$\mathcal{O}_{W_5}\mathcal{O}_{W_5} = f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + f_{xzz}\mathcal{O}_{W_3} + f_{wzz}\mathcal{O}_{W_4}. \tag{4.27j}$$

The quantum ring of $Gr(2,4)$ is

$$\begin{aligned}
\mathcal{O}_{W_2}\mathcal{O}_{W_2} &= f_{vvz}\mathcal{O}_{W_1} + f_{vvy}\mathcal{O}_{W_2} + (f_{vvw}+1)\mathcal{O}_{W_3} \\
&+ (f_{vvx}+1)\mathcal{O}_{W_4} + f_{vvv}\mathcal{O}_{W_5}, && (4.28a) \\
\mathcal{O}_{W_2}\mathcal{O}_{W_3} &= f_{vwz}\mathcal{O}_{W_1} + f_{vwy}\mathcal{O}_{W_2} + f_{vww}\mathcal{O}_{W_3} \\
&+ f_{vwx}\mathcal{O}_{W_4} + (f_{vvw}+1)\mathcal{O}_{W_5}, && (4.28b) \\
\mathcal{O}_{W_2}\mathcal{O}_{W_4} &= f_{vxz}\mathcal{O}_{W_1} + f_{vxy}\mathcal{O}_{W_2} + f_{vwx}\mathcal{O}_{W_3} + f_{vxx}\mathcal{O}_{W_4} + (f_{vvx}+1)\mathcal{O}_{W_5} && (4.28c) \\
\mathcal{O}_{W_2}\mathcal{O}_{W_5} &= f_{vyz}\mathcal{O}_{W_1} + f_{vyy}\mathcal{O}_{W_2} + f_{vwy}\mathcal{O}_{W_3} + f_{vxy}\mathcal{O}_{W_4} + f_{vvy}\mathcal{O}_{W_5}, && (4.28d) \\
\mathcal{O}_{W_2}\mathcal{O}_{W_6} &= f_{vzz}\mathcal{O}_{W_1} + f_{vyz}\mathcal{O}_{W_2} + f_{vwz}\mathcal{O}_{W_3} + f_{vxz}\mathcal{O}_{W_4} + f_{vvz}\mathcal{O}_{W_5}, && (4.28e) \\
\mathcal{O}_{W_3}\mathcal{O}_{W_3} &= f_{wwz}\mathcal{O}_{W_1} + f_{wwy}\mathcal{O}_{W_2} + f_{www}\mathcal{O}_{W_3} \\
&+ f_{wwx}\mathcal{O}_{W_4} + f_{vww}\mathcal{O}_{W_5} + \mathcal{O}_{W_6}, && (4.28f) \\
\mathcal{O}_{W_3}\mathcal{O}_{W_4} &= f_{wxz}\mathcal{O}_{W_1} + f_{wxy}\mathcal{O}_{W_2} + f_{wwx}\mathcal{O}_{W_3} + f_{wxx}\mathcal{O}_{W_4} + f_{vwx}\mathcal{O}_{W_5}, && (4.28g) \\
\mathcal{O}_{W_3}\mathcal{O}_{W_5} &= f_{wyz}\mathcal{O}_{W_1} + f_{wyy}\mathcal{O}_{W_2} + f_{wwy}\mathcal{O}_{W_3} + f_{wxy}\mathcal{O}_{W_4} + f_{vwy}\mathcal{O}_{W_5}, && (4.28h) \\
\mathcal{O}_{W_3}\mathcal{O}_{W_6} &= f_{wzz}\mathcal{O}_{W_1} + f_{wyz}\mathcal{O}_{W_2} + f_{wwz}\mathcal{O}_{W_3} + f_{wxz}\mathcal{O}_{W_4} + f_{vwz}\mathcal{O}_{W_5}, && (4.28i) \\
\mathcal{O}_{W_4}\mathcal{O}_{W_4} &= f_{xxz}\mathcal{O}_{W_1} + f_{xxy}\mathcal{O}_{W_2} + f_{wxx}\mathcal{O}_{W_3} \\
&+ f_{xxx}\mathcal{O}_{W_4} + f_{vxx}\mathcal{O}_{W_5} + \mathcal{O}_{W_6}, && (4.28j) \\
\mathcal{O}_{W_4}\mathcal{O}_{W_5} &= f_{xyz}\mathcal{O}_{W_1} + f_{xyy}\mathcal{O}_{W_2} + f_{wxy}\mathcal{O}_{W_3} + f_{xxy}\mathcal{O}_{W_4} + f_{vxy}\mathcal{O}_{W_5}, && (4.28k) \\
\mathcal{O}_{W_4}\mathcal{O}_{W_6} &= f_{xzz}\mathcal{O}_{W_1} + f_{xyz}\mathcal{O}_{W_2} + f_{wxz}\mathcal{O}_{W_3} + f_{xxz}\mathcal{O}_{W_4} + f_{vxz}\mathcal{O}_{W_5}, && (4.28l) \\
\mathcal{O}_{W_5}\mathcal{O}_{W_5} &= f_{yyz}\mathcal{O}_{W_1} + f_{yyy}\mathcal{O}_{W_2} + f_{wyy}\mathcal{O}_{W_3} + f_{xyy}\mathcal{O}_{W_4} + f_{vyy}\mathcal{O}_{W_5}, && (4.28m) \\
\mathcal{O}_{W_5}\mathcal{O}_{W_6} &= f_{yzz}\mathcal{O}_{W_1} + f_{yyz}\mathcal{O}_{W_2} + f_{wyz}\mathcal{O}_{W_3} + f_{xyz}\mathcal{O}_{W_4} + fvyz\mathcal{O}_{W_5}, && (4.28n) \\
\mathcal{O}_{W_6}\mathcal{O}_{W_6} &= f_{zzz}\mathcal{O}_{W_1} + f_{yzz}\mathcal{O}_{W_2} + f_{wzz}\mathcal{O}_{W_3} + f_{xzz}\mathcal{O}_{W_4} + f_{vzz}\mathcal{O}_{W_5}. && (4.28o)
\end{aligned}$$

Associativity condition (3.7) implies the free energy of $CP^3$ must satifying the following constraint equation

$$-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} = 0, \quad (4.29a)$$

$$-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0, \quad (4.29b)$$

$$f_{yzz} - f_{xxz}f_{yyy} + f_{xxy}f_{yyz} = 0, \quad (4.29c)$$

$$-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0, \tag{4.29d}$$

$$f_{zzz} - f_{xyz}^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0, \tag{4.29e}$$

$$f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0. \tag{4.29f}$$

For $CP^4$ there are 17 independent constraint equations. We just write down five of them which are enough to determine the correlation functions of $CP^4$

$$-f_{wwz} - f_{wwy}f_{wxx} + f_{wxx}^2 + 2f_{wwx}f_{xxx} - f_{www}f_{xxy} = 0, \tag{4.30a}$$

$$f_{wxy}^2 + f_{wwy}f_{wyy} + 2f_{wyz} - f_{wwx}f_{xyy} - f_{www}f_{yyy} = 0, \tag{4.30b}$$

$$f_{wxy}f_{wxz} + f_{wwz}f_{wyy} + f_{wzz} - f_{wwx}f_{xyz} - f_{www}f_{yyz} = 0, \tag{4.30c}$$

$$-f_{wxz}f_{xyy} + f_{wxy}f_{xyz} - f_{wwz}f_{yyy} + f_{wwy}f_{yyz} + f_{yzz} = 0, \tag{4.30d}$$

$$f_{wxy}f_{xxy} + f_{wwy}f_{xyy} - f_{wxx}f_{xyy} + f_{xyz} - f_{wwx}f_{yyy} = 0. \tag{4.30e}$$

For the case of $Gr(2,4)$ there are fifty-six independent equations. We also write down nine of them that determine the corelation functions of $Gr(2,4)$

$$\begin{aligned}&-f_{vvz} - f_{vvy}f_{vww} + f_{vww}^2 + f_{vwx}^2 + 2f_{vvw}f_{vwy}\\&-f_{vvw}f_{www} - f_{vvx}f_{wwx} - f_{vvv}f_{wwy} = 0,\end{aligned} \tag{4.31a}$$

$$\begin{aligned}&-f_{vvz} - f_{vvy}f_{vxx} + f_{vwx}^2 + f_{vxx}^2 + 2f_{vvx}f_{vxy}\\&-f_{vvw}f_{wxx} - f_{vvx}f_{xxx} - f_{vvv}f_{xxy} = 0,\end{aligned} \tag{4.31b}$$

$$\begin{aligned}&-f_{vxz} - f_{vww}f_{vxy} + f_{vwx}f_{vwy} - f_{vwx}f_{www} + f_{vww}f_{wwx}\\&-f_{vxx}f_{wwx} - f_{vvx}f_{wwy} + f_{vwx}f_{wxx} + f_{vvw}f_{wxy} = 0,\end{aligned} \tag{4.31c}$$

$$\begin{aligned}&-f_{wwz} - f_{xxz} - f_{vxx}f_{wwy} + f_{wwx}^2 + f_{wxx}^2 + 2f_{vwx}f_{wxy}\\&-f_{www}f_{wxx} - f_{wwx}f_{xxx} - f_{vww}f_{xxy} = 0,\end{aligned} \tag{4.31d}$$

$$\begin{aligned}&-f_{xyz} - f_{vxy}f_{wwy} + f_{wwx}f_{wwy} + f_{vwy}f_{wxy} + f_{wxx}f_{wxy}\\&-f_{www}f_{wxy} + f_{vwx}f_{wyy} - f_{wwx}f_{xxy} - f_{vww}f_{xyy} = 0,\end{aligned} \tag{4.31e}$$

$$\begin{aligned}&-f_{xzz} - f_{vxz}f_{wwy} + f_{wwx}f_{wwz} + f_{vwz}f_{wxy} + f_{wxx}f_{wxz}\\&-f_{www}f_{wxz} + f_{vwx}f_{wyz} - f_{wwx}f_{xxz} - f_{vww}f_{xyz} = 0,\end{aligned} \tag{4.31f}$$

$$\begin{aligned}&f_{wxy} + f_{vwy}f_{wwy} + f_{vxy}f_{wxy} + f_{vvy}f_{wyy}\\&-f_{vww}f_{wyy} - f_{vwx}f_{xyy} - f_{vvw}f_{yyy} = 0,\end{aligned} \tag{4.31g}$$

$$f_{wxz} - f_{vwy}f_{vxy} + f_{vwx}f_{vyy} + f_{vwy}f_{wwx} + f_{vxy}f_{wxx}$$
$$-f_{vwx}f_{wwy} + f_{vvy}f_{wxy} - f_{vxx}f_{wxy} - f_{vvx}f_{wyy} = 0, \tag{4.31h}$$

$$-f_{yyz} + f_{vwz}f_{wyy} - f_{vwy}f_{wyz} + f_{vxz}f_{xyy}$$
$$+f_{vvz}f_{yyy} - f_{vxy}f_{xyz} - f_{vvy}f_{yyz} = 0. \tag{4.31i}$$

Substituting the free energy (4.19–4.21) into (4.29),(4.30) and (4.31) one obtains the recursion relations of correlation functions. For $CP^3$ one has

$$2dN^d_{m+1} - N^d_m = \sum_{\substack{f+g=d \\ n+n'=m}} \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\left[ - \begin{pmatrix} 4d-2m-3 \\ 4f-2n-2 \end{pmatrix} fN^f_n N^g_{n'} + \begin{pmatrix} 4d-2m-3 \\ 4f-2n-3 \end{pmatrix} g^3 N^f_n N^g_{n'} \right], \tag{4.32a}$$

$$dN^d_{m+2} - N^d_{m+1} = \sum_{\substack{f+g=d \\ n+n'=m}} \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\left[ \begin{pmatrix} 4d-2m-4 \\ 4f-2n-4 \end{pmatrix} g^3 N^f_{n+1} N^g_{n'} - \begin{pmatrix} 4d-2m-4 \\ 4f-2n-2 \end{pmatrix} fg^2 N^f_n N^g_{n'+1} \right], \tag{4.32b}$$

$$N^d_{m+2} = \sum_{\substack{f+g=d \\ n+n'=m}} \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\left[ \begin{pmatrix} 4d-2m-5 \\ 4f-2n-2 \end{pmatrix} f^2 N^f_{n+1} N^g_{n'} - \begin{pmatrix} 4d-2m-5 \\ 4f-2n-1 \end{pmatrix} f^2 N^f_n N^g_{n'+1} \right]. \tag{4.32c}$$

For the case of $CP^4$ the recursion relations read as follows

$$d^2 N^d_{m_1,m_2+1} - 2dN^d_{m_1,m_2} + N^d_{m_1,m_2}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \begin{pmatrix} m_2 \\ n_2 \end{pmatrix}$$

$$\left[ - \begin{pmatrix} 5d-2m_1-3m_2-2 \\ 5f-2n_1-3n_2-1 \end{pmatrix} f^2 g N^f_{n_1+1,n_2} N^g_{n_1',n_2'} \right.$$

$$+ \begin{pmatrix} 5d-2m_2-3m_3-2 \\ 5f-2n_1-3n_2-1 \end{pmatrix} fg N^f_{n_1,n_2} N^g_{n_1',n_2'}$$

$$+2 \begin{pmatrix} 5d-2m_1-3m_2-2 \\ 5f-2n_1-3n_2 \end{pmatrix} f^2 g N^f_{n_1,n_2} N^g_{n_1'+1,n_2'}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-2\\5f-2n_1-3n_2\end{array}\right)f^2N^f_{n_1,n_2}N^g_{n'_1,n'_2}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-2\\5f-2n_1-3n_2+1\end{array}\right)f^3N^f_{n_1,n_2}N^g_{n'_1,n'_2+1}\Big], \tag{4.33a}$$

$$N^d_{m_1+1,m_2+1}-dN^d_{m_1,m_2+2}$$

$$=\sum_{\substack{f+g=d\\n_1+n'_1=m_1,n_2+n'_2=m_2}}\left(\begin{array}{c}m_1\\n_1\end{array}\right)\left(\begin{array}{c}m_2\\n_2\end{array}\right)$$

$$\Big[\left(\begin{array}{c}5d-2m_1-3m_2-5\\5f-2n_1-3n_2-2\end{array}\right)fgN^f_{n_1+1,n_2}N^g_{n'_1,n'_2+1}$$

$$+\left(\begin{array}{c}5d-2m_2-3m_3-5\\5f-2n_1-3n_2-2\end{array}\right)f^2gN^f_{n_1,n_2+1}N^g_{n'_1+2,n'_2}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-5\\5f-2n_1-3n_2\end{array}\right)f^2N^f_{n_1,n_2}N^g_{n'_1+1,n'_2+1}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-5\\5f-2n_1-3n_2+1\end{array}\right)f^3N^f_{n_1,n_2}N^g_{n'_1+2,n'_2+1}\Big], \tag{4.33b}$$

$$N^d_{m_1+2,m_2}-2dN^d_{m_1+1,m_2+1}$$

$$=\sum_{\substack{f+g=d\\n_1+n'_1=m_1,n_2+n'_2=m_2}}\left(\begin{array}{c}m_1\\n_1\end{array}\right)\left(\begin{array}{c}m_2\\n_2\end{array}\right)$$

$$\Big[\left(\begin{array}{c}5d-2m_1-3m_2-4\\5f-2n_1-3n_2-2\end{array}\right)fgN^f_{n_1+1,n_2}N^g_{n'_1+1,n'_2}$$

$$+\left(\begin{array}{c}5d-2m_2-3m_3-4\\5f-2n_1-3n_2-1\end{array}\right)f^2gN^f_{n_1+1,n_2}N^g_{n'_1+2,n'_2}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-4\\5f-2n_1-3n_2\end{array}\right)f^2N^f_{n_1,n_2}N^g_{n'_1+2,n'_2}$$

$$-\left(\begin{array}{c}5d-2m_1-3m_2-4\\5f-2n_1-3n_2+1\end{array}\right)f^3N^f_{n_1,n_2}N^g_{n'_1+3,n'_2}\Big], \tag{4.33c}$$

$$N^d_{m_1+1,m_2+2}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Big[\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2-3} f N^f_{n_1,n_2+1} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_2-3m_3-7}{5f-2n_1-3n_2-2} f N^f_{n_1+1,n_2} N^g_{n_1'+1,n_2'+1}$$

$$+\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2+1} N^g_{n_1'+3,n_2'}$$

$$-\binom{5d-2m_1-3m_2-7}{5f-2n_1-3n_2-1} f^2 N^f_{n_1+1,n_2} N^g_{n_1'+2,n_2'+1}\Big]. \tag{4.33d}$$

For the case of $Gr(2,4)$ the recursion relation of becomes

$$N^d_{m_1+3,m_2} - N^d_{m_1+1,m_2+1}$$

$$= \sum_{\substack{f+g=d \\ n_1+n_1'=m_1, n_2+n_2'=m_2}} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

$$\Big[\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2-2} f N^f_{n_1+1,n_2} N^g_{n_1'+1,n_2'}$$

$$+\binom{5d-2m_2-3m_3-5}{5f-2n_1-3n_2-1} f^2 N^f_{n_1+1,n_2} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2-1} f N^f_{n_1,n_2} N^g_{n_1'+2,n_2'}$$

$$-\binom{5d-2m_1-3m_2-5}{5f-2n_1-3n_2} f^2 N^f_{n_1,n_2} N^g_{n_1'+3,n_2'}\Big], \tag{4.34a}$$

$$d^2 N^d_{m_1,m_2,m_3+1} - 2d N^d_{m_1,m_2+1,m_3} + N^d_{m_1,m_2,m_3} + N^d_{m_1+1,m_2,m_3}$$

$$= \sum_{\substack{f+g=d, n_1+n_1'=m_1 \\ n_2+n_2'=m_2, n_3+n_3'=m_3}} \binom{m_1}{n_1}\binom{m_2}{n_2}\binom{m_3}{n_3}$$

$$\Big[-\binom{4d-m_1-2m_2-3m_3-2}{4f-n_1-2n_2-3n_3-1} f^2 g N^f_{n_1,n_2+1,n_3} N^g_{n_1',n_2',n_3'}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fgN^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fgN^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2,n'_3}$$

$$+2\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2gN^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3+1 \end{pmatrix} f^3N^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+1,n'_3}\Big], \tag{4.34b}$$

$$d^2N^d_{m_1,m_2,m_3+1}-2dN^d_{m_1+1,m_2+1,m_3}+N^d_{m_1+2,m_2,m_3}+N^d_{m_1+3,m_2,m_3}$$

$$=\sum_{\substack{f+g=d,n_1+n'_1=m_1\\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}\begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fgN^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f^2gN^f_{n_1,n_2+1,n_3}N^g_{n'_1+2,n'_2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fgN^f_{n_1+2,n_2,n_3}N^g_{n'_1+2,n'_2,n'_3}$$

$$+2\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2gN^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1,n_2,n_3}N^g_{n'_1+2,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1+1,n_2,n_3}N^g_{n'_1+3,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-2 \\ 4f-n_1-2n_2-3n_3+1 \end{pmatrix} f^3 N^f_{n_1,n_2,n_3} N^g_{n'_1+2,n'_2+1,n'_3}\Big], \tag{4.34c}$$

$$dN^d_{m_1+1,m_2,m_3+1} + N^d_{m_1,m_2+1,m_3} - N^d_{m_1+1,m_2+1,m_3}$$

$$= \sum_{\substack{f+g=d,n_1+n'_1=m_1 \\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \begin{pmatrix} m_2 \\ n_2 \end{pmatrix} \begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fg N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fg N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1+2,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2 N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1+1,n_2,n_3} N^g_{n'_1+2,n'_2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-3 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2 N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2+1,n'_3}\Big], \tag{4.34d}$$

$$N^d_{m_1,m_2,m_3+1} + N^d_{m_1+2,m_2,m_3+1}$$

$$= \sum_{\substack{f+g=d,n_1+n'_1=m_1 \\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \begin{pmatrix} m_2 \\ n_2 \end{pmatrix} \begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+2,n_2,n_3}N^g_{n_1',n_2'+1,n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1,n_2,n_3}N^g_{n_1'+2,n_2',n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+2,n_2,n_3}N^g_{n_1'+2,n_2',n_3'}$$

$$+2\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+1,n_2,n_3}N^g_{n_1'+1,n_2'+1,n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3}N^g_{n_1'+3,n_2',n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1,n_2,n_3}N^g_{n_1'+2,n_2'+1,n_3'}\Big], \qquad (4.34e)$$

$$N^d_{m_1+1,m_2+1,m_3+1}$$

$$= \sum_{\substack{f+g=d,n_1+n_1'=m_1 \\ n_2+n_2'=m_2,n_3+n_3'=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}\begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1+1,n_2+1,n_3}N^g_{n_1',n_2'+1,n_3'}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3}N^g_{n_1',n_2'+1,n_3'}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1,n_2+1,n_3}N^g_{n_1'+1,n_2'+1,n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1,n_2,n_3}N^g_{n_1'+1,n_2'+1,n_3'}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+2,n_2,n_3}N^g_{n_1'+1,n_2'+1,n_3'}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+1,n_2,n_3}N^g_{n_1',n_2'+2,n_3'}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3} N^g_{n'_1+2,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2+2,n'_3}\Big], \tag{4.34f}$$

$$N^d_{m_1+1,m_2,m_3+2}$$

$$= \sum_{\substack{f+g=d,n_1+n'_1=m_1 \\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \begin{pmatrix} m_2 \\ n_2 \end{pmatrix} \begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[-\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-3 \end{pmatrix} f N^f_{n_1+1,n_2,n_3+1} N^g_{n'_1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2,n'_3+1}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-3 \end{pmatrix} f N^f_{n_1,n_2,n_3+1} N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3+1}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3} N^g_{n'_1+1,n'_2,n'_3+1}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1+1,n_2,n_3} N^g_{n'_1,n'_2+1,n'_3+1}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} N^f_{n_1+1,n_2,n_3} N^g_{n'_1+2,n'_2,n'_3+1}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-6 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f N^f_{n_1,n_2,n_3} N^g_{n'_1+1,n'_2+1,n'_3+1}\Big], \tag{4.34g}$$

$$N^d_{m_1,m_2+3,m_3} - N^d_{m_1,m_2+1,m_3+1}$$

$$= \sum_{\substack{f+g=d,n_1+n'_1=m_1 \\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \begin{pmatrix} m_2 \\ n_2 \end{pmatrix} \begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1+1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f^2N^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+1,n_2,n_3}N^g_{n'_1+1,n'_2+2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-5 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}\Big], \tag{4.34h}$$

$$N^d_{m_1,m_2+2,m_3}-N^d_{m_1+1,m_2,m_3+1}$$

$$=\sum_{\substack{f+g=d,n_1+n'_1=m_1\\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}\begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fgN^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fgN^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+1,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1+1,n_2+1,n_3}N^g_{n'_1+2,n'_2,n'_3}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f^2N^f_{n_1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} fN^f_{n_1+2,n_2,n_3}N^g_{n'_1+1,n'_2+1,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-4 \\ 4f-n_1-2n_2-3n_3 \end{pmatrix} f^2N^f_{n_1+1,n_2,n_3}N^g_{n'_1,n'_2+2,n'_3}\Big], \tag{4.34i}$$

$$N^d_{m_1,m_2+1,m_3+2}$$

$$= \sum_{\substack{f+g=d,n_1+n'_1=m_1 \\ n_2+n'_2=m_2,n_3+n'_3=m_3}} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}\begin{pmatrix} m_3 \\ n_3 \end{pmatrix}$$

$$\Big[\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-3 \end{pmatrix} fN^f_{n_1,n_2,n_3+1}N^g_{n'_1,n'_2+2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+1,n'_3+1}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-3 \end{pmatrix} fN^f_{n_1+1,n_2,n_3+1}N^g_{n'_1+1,n'_2+2,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} fN^f_{n_1+1,n_2+1,n_3}N^g_{n'_1+1,n'_2+1,n'_3+1}$$

$$+\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-2 \end{pmatrix} f^2N^f_{n_1,n_2,n_3+1}N^g_{n'_1,n'_2+3,n'_3}$$

$$-\begin{pmatrix} 4d-m_1-2m_2-3m_3-7 \\ 4f-n_1-2n_2-3n_3-1 \end{pmatrix} f^2N^f_{n_1,n_2+1,n_3}N^g_{n'_1,n'_2+2,n'_3+1}\Big]. \tag{4.34j}$$

In there recursion relations, d, f, and g are all greater or equal to one. So when d equals one, r,h,s of these equations vanish since g+f $\geq$ 2. Then we have a set of linear relations for $N^1_*$'s. We can use these linear relations to determine all the the $<\mathcal{O}_{W_4}\mathcal{O}_{W_4}>$=1 for $CP^3$, $<\mathcal{O}_{W_5}\mathcal{O}_{W_5}>$=1 for $CP^4$ and $<\mathcal{O}_{W_5}\mathcal{O}_{W_6}>$=1 for $Gr(2,4)$. Then we put these degree 1 correlation functions to the r,h,s of (4.35),(4.36),(4.37) and obtain linear relations for $N^2_*$'s. This time, these linear relations thoroughly determine them. For higher degree, the process is the same as d=2 case. We observe that recursion relations we have witten down are suffcient for determination. We checked the compatiable condition in the case of d $\leq$ 4. It seems that the over determined system of WDVV equation work well in all degrees of maps in the case of $CP^3$,$CP^4$ and $Gr(2,4)$. The intersection numbers of moduli space of d $\leq$ 4 are given in the following tables.

Table 4: **D=1** $CP^3$

| | | |
|---|---|---|
| $N_0 = 2$ | $N_1 = 1$ | $N_2 = 1$ |

Table 5: D=2 $CP^3$

| | | | | |
|---|---|---|---|---|
| $N_0 = 92$ | $N_1 = 18$ | $N_2 = 4$ | $N_3 = 1$ | $N_4 = 0$ |

Table 6: D=3 $CP^3$

| | | | | | | |
|---|---|---|---|---|---|---|
| $N_0 = 80160$ | $N_1 = 9864$ | $N_2 = 1312$ | $N_3 = 190$ | $N_4 = 30$ | $N_5 = 5$ | $N_6 = 1$ |

Table 7: D=4 $CP^3$

| | | | | |
|---|---|---|---|---|
| $N_0 = 383306880$ | $N_1 = 34382544$ | $N_2 = 3259680$ | $N_3 = 327888$ | $N_4 = 35104$ |
| $N_5 = 4000$ | $N_6 = 480$ | $N_7 = 58$ | $N_8 = 4$ | |

Table 8: D=5 $CP^3$

| | | | |
|---|---|---|---|
| $N_0 = 6089786376960$ | $N_1 = 429750191232$ | $N_2 = 31658432256$ | $N_3 = 2440235712$ |
| $N_4 = 197240400$ | $N_5 = 16744080$ | $N_6 = 1492616$ | $N_7 = 139098$ |
| $N_8 = 13354$ | $N_9 = 1265$ | $N_{10} = 105$ | |

Table 9: D=6 $CP^3$

| | | |
|---|---|---|
| $N_0 = 244274488980962304$ | $N_1 = 14207926965714432$ | $N_2 = 855909223176192$ |
| $N_3 = 53486265350784$ | $N_4 = 3472451647488$ | $N_5 = 234526910784$ |
| $N_6 = 16492503552$ | $N_7 = 1207260512$ | $N_8 = 91797312$ |
| $N_9 = 7200416$ | $N_{10} = 573312$ | $N_{11} = 44416$ |
| $N_{12} = 2576$ | | |

Table 10: D=1 $CP^4$

| | | | | | | |
|---|---|---|---|---|---|---|
| $N_{00} = 5$ | $N_{10} = 3$ | $N_{20} = 2$ | $N_{30} = 1$ | $N_{01} = 1$ | $N_{11} = 1$ | $N_{02} = 1$ |

Table 11: D=2 $CP^4$

| | | | | | |
|---|---|---|---|---|---|
| $N_{00} = 6620$ | $N_{10} = 1734$ | $N_{20} = 473$ | $N_{30} = 132$ | $N_{40} = 36$ | $N_{50} = 10$ |
| $N_{01} = 219$ | $N_{11} = 67$ | $N_{21} = 21$ | $N_{31} = 6$ | $N_{41} = 2$ | |
| $N_{02} = 11$ | $N_{12} = 4$ | $N_{22} = 1$ | | | |
| $N_{03} = 1$ | $N_{13} = 0$ | | | | |

Table 12: D=3 $CP^4$

| | | | | |
|---|---|---|---|---|
| $N_{00} = 213709980$ | $N_{01} = 2770596$ | $N_{02} = 45954$ | $N_{03} = 1011$ | $N_{04} = 30$ |
| $N_{05} = 0$ | $N_{10} = 35806494$ | $N_{11} = 511012$ | $N_{12} = 9386$ | $N_{13} = 225$ |
| $N_{14} = 5$ | $N_{20} = 6165822$ | $N_{21} = 96548$ | $N_{22} = 1931$ | $N_{23} = 45$ |
| $N_{24} = 1$ | $N_{30} = 1085892$ | $N_{31} = 18469$ | $N_{32} = 385$ | $N_{33} = 9$ |
| $N_{40} = 194024$ | $N_{41} = 3512$ | $N_{42} = 76$ | | |
| $N_{50} = 34780$ | $N_{51} = 664$ | $N_{52} = 16$ | | |
| $N_{60} = 6216$ | $N_{61} = 128$ | | | |
| $N_{70} = 1108$ | | | | |
| $N_{80} = 188$ | | | | |

Table 13: D=4 $CP^4$

| | | | |
|---|---|---|---|
| $N_{00} = 477234479050600$ | $N_{01} = 327439797532$ | $N_{02} = 2679044142$ | $N_{03} = 26578256$ |
| $N_{04} = 324764$ | $N_{05} = 4830$ | $N_{06} = 61$ | $N_{07} = 1$ |
| $N_{10} = 5876564125104$ | $N_{11} = 43242657488$ | $N_{12} = 380720598$ | $N_{13} = 4063860$ |
| $N_{14} = 52507$ | $N_{15} = 732$ | $N_{16} = 9$ | |
| $N_{20} = 738764469204$ | $N_{21} = 5823161346$ | $N_{22} = 54948346$ | $N_{23} = 622980$ |
| $N_{24} = 8133$ | $N_{25} = 107$ | | |
| $N_{30} = 94605276228$ | $N_{31} = 796460052$ | $N_{32} = 7990720$ | $N_{33} = 94104$ |
| $N_{34} = 1218$ | $N_{35} = 14$ | | |
| $N_{40} = 12302188692$ | $N_{41} = 110031632$ | $N_{42} = 1159218$ | $N_{43} = 13962$ |
| $N_{44} = 178$ | | | |
| $N_{50} = 1617593360$ | $N_{51} = 15251816$ | $N_{52} = 166936$ | $N_{53} = 2056$ |
| $N_{60} = 213984472$ | $N_{61} = 2110864$ | $N_{62} = 23968$ | $N_{63} = 320$ |
| $N_{70} = 28346212$ | $N_{71} = 291632$ | $N_{72} = 3516$ | |
| $N_{80} = 3748804$ | $N_{81} = 40492$ | | |
| $N_{90} = 343260$ | $N_{91} = 5552$ | | |
| $N_{100} = 63740$ | | | |

Table 14: D=1 Gr(2,4)

| | | | | | |
|---|---|---|---|---|---|
| $N_{000}=0$ | $N_{100}=0$ | $N_{200}=1$ | $N_{300}=1$ | $N_{400}=0$ | $N_{500}=0$ |
| $N_{001}=0$ | $N_{101}=1$ | $N_{201}=0$ | | | |
| $N_{010}=0$ | $N_{110}=1$ | $N_{210}=1$ | $N_{310}=0$ | | |
| $N_{011}=1$ | | | | | |
| $N_{020}=1$ | $N_{120}=1$ | | | | |

Table 15: D=2 Gr(2,4)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $N_{000}=2$ | $N_{100}=6$ | $N_{200}=18$ | $N_{300}=34$ | $N_{400}=42$ | $N_{500}=42$ | $N_{600}=34$ | $N_{700}=18$ |
| $N_{800}=6$ | $N_{900}=2$ | | | | | | |
| $N_{001}=1$ | $N_{101}=3$ | $N_{201}=5$ | $N_{301}=5$ | $N_{401}=5$ | $N_{501}=3$ | $N_{601}=1$ | |
| $N_{002}=1$ | $N_{102}=1$ | $N_{202}=1$ | $N_{302}=1$ | | | | |
| $N_{003}=1$ | | | | | | | |
| $N_{010}=3$ | $N_{110}=9$ | $N_{210}=17$ | $N_{310}=21$ | $N_{410}=21$ | $N_{510}=17$ | $N_{610}=9$ | $N_{710}=3$ |
| $N_{011}=2$ | $N_{111}=3$ | $N_{211}=3$ | $N_{311}=3$ | $N_{411}=2$ | | | |
| $N_{012}=1$ | $N_{112}=1$ | | | | | | |
| $N_{020}=5$ | $N_{120}=9$ | $N_{220}=11$ | $N_{320}=11$ | $N_{420}=9$ | $N_{520}=5$ | | |
| $N_{021}=2$ | $N_{121}=2$ | $N_{221}=2$ | | | | | |
| $N_{030}=5$ | $N_{130}=6$ | $N_{230}=6$ | $N_{330}=5$ | | | | |
| $N_{031}=1$ | | | | | | | |
| $N_{040}=3$ | $N_{140}=3$ | | | | | | |

Table 16: D=3 Gr(2,4)

| | | | | |
|---|---|---|---|---|
| $N_{000} = 504$ | $N_{001} = 100$ | $N_{002} = 25$ | $N_{003} = 6$ | $N_{004} = 2$ |
| $N_{100} = 1824$ | $N_{101} = 307$ | $N_{102} = 55$ | $N_{103} = 9$ | $N_{104} = 2$ |
| $N_{200} = 5159$ | $N_{201} = 676$ | $N_{202} = 83$ | $N_{203} = 10$ | |
| $N_{300} = 11319$ | $N_{301} = 1109$ | $N_{302} = 101$ | $N_{303} = 9$ | |
| $N_{400} = 19512$ | $N_{401} = 1460$ | $N_{402} = 101$ | $N_{403} = 6$ | |
| $N_{500} = 27472$ | $N_{501} = 1605$ | $N_{502} = 83$ | | |
| $N_{600} = 32517$ | $N_{601} = 1460$ | $N_{602} = 55$ | | |
| $N_{700} = 32517$ | $N_{701} = 1109$ | $N_{702} = 25$ | | |
| $N_{800} = 27472$ | $N_{801} = 676$ | | | |
| $N_{900} = 19512$ | $N_{901} = 307$ | | | |
| $N_{1000} = 11319$ | $N_{1001} = 100$ | | | |
| $N_{1100} = 5159$ | | | | |
| $N_{1200} = 1824$ | | | | |
| $N_{1300} = 504$ | | | | |
| $N_{010} = 538$ | $N_{011} = 109$ | $N_{012} = 23$ | $N_{013} = 5$ | |
| $N_{110} = 1603$ | $N_{111} = 246$ | $N_{112} = 35$ | $N_{113} = 5$ | |
| $N_{210} = 3607$ | $N_{211} = 403$ | $N_{212} = 42$ | $N_{213} = 5$ | |
| $N_{310} = 6278$ | $N_{311} = 528$ | $N_{312} = 42$ | | |
| $N_{410} = 8864$ | $N_{411} = 579$ | $N_{412} = 35$ | | |
| $N_{510} = 10499$ | $N_{511} = 528$ | $N_{512} = 23$ | | |
| $N_{610} = 10499$ | $N_{611} = 403$ | | | |
| $N_{710} = 8864$ | $N_{711} = 246$ | | | |
| $N_{810} = 6278$ | $N_{811} = 109$ | | | |
| $N_{910} = 3609$ | | | | |
| $N_{1010} = 1603$ | | | | |
| $N_{1110} = 538$ | | | | |
| $N_{020} = 523$ | $N_{021} = 94$ | $N_{022} = 16$ | $N_{023} = 2$ | |
| $N_{120} = 1203$ | $N_{121} = 153$ | $N_{122} = 18$ | | |
| $N_{220} = 2100$ | $N_{221} = 198$ | $N_{222} = 18$ | | |
| $N_{320} = 2960$ | $N_{321} = 216$ | $N_{322} = 16$ | | |
| $N_{420} = 3501$ | $N_{421} = 198$ | | | |
| $N_{520} = 3501$ | $N_{521} = 153$ | | | |
| $N_{620} = 2960$ | $N_{621} = 94$ | | | |
| $N_{720} = 2100$ | | | | |
| $N_{820} = 1203$ | | | | |
| $N_{920} = 523$ | | | | |
| $N_{030} = 420$ | $N_{031} = 61$ | $N_{032} = 7$ | | |
| $N_{130} = 729$ | $N_{131} = 76$ | $N_{132} = 7$ | | |
| $N_{230} = 1019$ | $N_{231} = 82$ | | | |
| $N_{330} = 1200$ | $N_{331} = 76$ | | | |
| $N_{430} = 1200$ | $N_{431} = 61$ | | | |
| $N_{530} = 1019$ | | | | |
| $N_{630} = 729$ | | | | |
| $N_{730} = 420$ | | | | |
| $N_{040} = 262$ | $N_{041} = 28$ | | | |
| $N_{140} = 358$ | $N_{141} = 30$ | | | |
| $N_{240} = 418$ | $N_{241} = 28$ | | | |
| $N_{340} = 418$ | | | | |
| $N_{440} = 358$ | | | | |
| $N_{540} = 262$ | | | | |
| $N_{050} = 124$ | $N_{051} = 10$ | | | |
| $N_{150} = 144$ | | | | |
| $N_{250} = 144$ | | | | |
| $N_{350} = 124$ | | | | |
| $N_{060} = 48$ | | | | |
| $N_{160} = 48$ | | | | |

Table 17: D=4 Gr(2,4)

| | | | | | |
|---|---|---|---|---|---|
| $N_{000} = 1044120$ | $N_{001} = 93726$ | $N_{002} = 9970$ | $N_{003} = 1170$ | $N_{004} = 138$ | $N_{005} = 20$ |
| $N_{100} = 3094440$ | $N_{101} = 251402$ | $N_{102} = 22570$ | $N_{103} = 2058$ | $N_{104} = 190$ | $N_{105} = 20$ |
| $N_{200} = 8093840$ | $N_{201} = 570998$ | $N_{202} = 4179$ | $N_{203} = 2998$ | $N_{204} = 214$ | $N_{205} = 20$ |
| $N_{300} = 18245976$ | $N_{301} = 1086890$ | $N_{302} = 64434$ | $N_{303} = 3690$ | $N_{304} = 214$ | |
| $N_{400} = 35219976$ | $N_{401} = 1752446$ | $N_{402} = 84818$ | $N_{403} = 3942$ | $N_{404} = 190$ | |
| $N_{500} = 58571280$ | $N_{501} = 2434530$ | $N_{502} = 96894$ | $N_{503} = 3690$ | $N_{504} = 138$ | |
| $N_{600} = 84843440$ | $N_{601} = 2951174$ | $N_{602} = 96894$ | $N_{603} = 2998$ | | |
| $N_{700} = 108066120$ | $N_{701} = 3143726$ | $N_{702} = 84818$ | $N_{703} = 2058$ | | |
| $N_{800} = 121770480$ | $N_{801} = 2951174$ | $N_{802} = 64434$ | $N_{803} = 1170$ | | |
| $N_{900} = 121770480$ | $N_{901} = 2434530$ | $N_{902} = 41794$ | | | |
| $N_{1000} = 108066120$ | $N_{1001} = 1752446$ | $N_{1002} = 22570$ | | | |
| $N_{1100} = 84843440$ | $N_{1101} = 1086890$ | $N_{1102} = 9970$ | | | |
| $N_{1200} = 58571280$ | $N_{1201} = 570998$ | | | | |
| $N_{1300} = 35219976$ | $N_{1301} = 251402$ | | | | |
| $N_{1400} = 18245976$ | $N_{1401} = 93726$ | | | | |
| $N_{1500} = 8093840$ | | | | | |
| $N_{1600} = 3094440$ | | | | | |
| $N_{1700} = 1044120$ | | | | | |
| $N_{010} = 692760$ | $N_{011} = 63904$ | $N_{012} = 6528$ | $N_{013} = 675$ | $N_{014} = 74$ | $N_{015} = 6$ |
| $N_{110} = 1852184$ | $N_{111} = 147070$ | $N_{112} = 12060$ | $N_{113} = 976$ | $N_{114} = 80$ | |
| $N_{210} = 4249660$ | $N_{211} = 281764$ | $N_{212} = 18506$ | $N_{213} = 1181$ | $N_{214} = 80$ | |
| $N_{310} = 8297556$ | $N_{311} = 454858$ | $N_{312} = 24196$ | $N_{313} = 1254$ | $N_{314} = 74$ | |
| $N_{410} = 13886500$ | $N_{411} = 631136$ | $N_{412} = 27526$ | $N_{413} = 1181$ | | |
| $N_{510} = 20177804$ | $N_{511} = 764000$ | $N_{512} = 27526$ | $N_{513} = 976$ | | |
| $N_{610} = 25736664$ | $N_{611} = 813396$ | $N_{612} = 24196$ | $N_{613} = 675$ | | |
| $N_{710} = 29015656$ | $N_{711} = 764000$ | $N_{712} = 18506$ | | | |
| $N_{810} = 29015656$ | $N_{811} = 631136$ | $N_{812} = 12060$ | | | |
| $N_{910} = 25736664$ | $N_{911} = 454858$ | $N_{912} = 6528$ | | | |
| $N_{1010} = 20177804$ | $N_{1011} = 281764$ | | | | |
| $N_{1110} = 13886500$ | $N_{1111} = 147070$ | | | | |
| $N_{1210} = 8297556$ | $N_{1211} = 63904$ | | | | |
| $N_{1310} = 4249660$ | | | | | |
| $N_{1410} = 1852184$ | | | | | |
| $N_{1510} = 692760$ | | | | | |
| $N_{020} = 440638$ | $N_{021} = 39460$ | $N_{022} = 3624$ | $N_{023} = 332$ | $N_{024} = 26$ | |
| $N_{120} = 1025894$ | $N_{121} = 75712$ | $N_{122} = 5512$ | $N_{123} = 338$ | $N_{124} = 26$ | |
| $N_{220} = 2019894$ | $N_{221} = 121884$ | $N_{222} = 7112$ | $N_{223} = 408$ | | |
| $N_{320} = 3391958$ | $N_{321} = 168332$ | $N_{322} = 8032$ | $N_{323} = 338$ | | |
| $N_{420} = 4932358$ | $N_{421} = 203048$ | $N_{422} = 8032$ | $N_{423} = 332$ | | |
| $N_{520} = 6290046$ | $N_{521} = 215904$ | $N_{522} = 7112$ | | | |
| $N_{620} = 7089646$ | $N_{621} = 203048$ | $N_{622} = 5512$ | | | |
| $N_{720} = 7089646$ | $N_{721} = 168332$ | $N_{722} = 3624$ | | | |
| $N_{820} = 6290046$ | $N_{821} = 121884$ | | | | |
| $N_{920} = 4932358$ | $N_{921} = 75712$ | | | | |
| $N_{1020} = 3391958$ | $N_{1021} = 39460$ | | | | |
| $N_{1120} = 2019894$ | | | | | |
| $N_{1220} = 1025894$ | | | | | |
| $N_{1320} = 440638$ | | | | | |

Table 18: D=4 Gr(2,4)

| | | | | | |
|---|---|---|---|---|---|
| $N_{030} = 256946$ | $N_{031} = 21072$ | $N_{032} = 1695$ | $N_{033} = 121$ | | |
| $N_{130} = 508026$ | $N_{131} = 33665$ | $N_{132} = 2131$ | $N_{133} = 126$ | | |
| $N_{230} = 852818$ | $N_{231} = 46042$ | $N_{232} = 2379$ | $N_{233} = 121$ | | |
| $N_{330} = 1237234$ | $N_{331} = 55181$ | $N_{332} = 2379$ | | | |
| $N_{430} = 1574370$ | $N_{431} = 58548$ | $N_{432} = 2131$ | | | |
| $N_{530} = 1772374$ | $N_{531} = 55181$ | $N_{532} = 1695$ | | | |
| $N_{630} = 1772374$ | $N_{631} = 46042$ | | | | |
| $N_{730} = 1574370$ | $N_{731} = 33665$ | | | | |
| $N_{830} = 1237234$ | $N_{831} = 21072$ | | | | |
| $N_{930} = 852818$ | | | | | |
| $N_{1030} = 508026$ | | | | | |
| $N_{1130} = 256946$ | | | | | |
| $N_{040} = 131874$ | $N_{041} = 9540$ | $N_{042} = 626$ | $N_{043} = 36$ | | |
| $N_{140} = 220250$ | $N_{141} = 12808$ | $N_{142} = 690$ | | | |
| $N_{240} = 317466$ | $N_{241} = 15196$ | $N_{242} = 690$ | | | |
| $N_{340} = 402090$ | $N_{341} = 16072$ | $N_{342} = 626$ | | | |
| $N_{440} = 451610$ | $N_{441} = 15196$ | | | | |
| $N_{540} = 451610$ | $N_{541} = 12808$ | | | | |
| $N_{640} = 402090$ | $N_{641} = 9540$ | | | | |
| $N_{740} = 317466$ | | | | | |
| $N_{840} = 220250$ | | | | | |
| $N_{940} = 131874$ | | | | | |
| $N_{050} = 58170$ | $N_{051} = 3544$ | $N_{052} = 190$ | | | |
| $N_{150} = 82790$ | $N_{151} = 4156$ | $N_{152} = 190$ | | | |
| $N_{250} = 104070$ | $N_{251} = 4380$ | | | | |
| $N_{350} = 116486$ | $N_{351} = 4156$ | | | | |
| $N_{450} = 116486$ | $N_{451} = 3544$ | | | | |
| $N_{550} = 104070$ | | | | | |
| $N_{650} = 82790$ | | | | | |
| $N_{750} = 58170$ | | | | | |
| $N_{060} = 21638$ | $N_{061} = 1104$ | | | | |
| $N_{160} = 26958$ | $N_{161} = 1160$ | | | | |
| $N_{260} = 30062$ | $N_{261} = 1104$ | | | | |
| $N_{360} = 30062$ | | | | | |
| $N_{460} = 26958$ | | | | | |
| $N_{560} = 21638$ | | | | | |
| $N_{070} = 6888$ | $N_{071} = 290$ | | | | |
| $N_{170} = 7664$ | | | | | |
| $N_{270} = 7664$ | | | | | |
| $N_{370} = 6888$ | | | | | |
| $N_{080} = 1916$ | | | | | |
| $N_{180} = 1916$ | | | | | |

# 5 Conclusion

We have shown in this paper that the free energy of topological sigma models on $CP^3$,$CP^4$ and $Gr(2,4)$ (coupled to gravity but on small phase space) can actually be evaluated using DWVV equation, some properties of topological field theory and one initial condition($\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle = 1$ for $CP^3$,$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle = 1$ for $CP^4$ and $\langle \mathcal{O}_{W_5}\mathcal{O}_{W_6}\rangle = 1$ for $Gr(2,4)$). Although in $CP^3$ and $CP^4$ case where classical cohomology ring of target space $M$ is generated by $H^2(M)$, this fact was already shown by Kontsevich and Manin, we find this strategy also work for $Gr(2,4)$. We also observe that for $CP^3$ and $CP^4$, expansion coefficients of the free energy acctually counts the number of rational curves passing through $PD(W)$ of insertion operator $\mathcal{O}_W$ in $d = 1$ case. (see Appendix A) From (2.9) we see the contribution of one rational curve satisfying "passing through condition" to correlation function is the product of intersection numbers between the rational curve and corresponding Poincaré duals, however, it seems this factor equals 1, unless $W$ is the Kähler form. If not so, most of correlation functions in degree $d$ sector have to be devisible by $d$, but our results do not support this speculation. In $Gr(2,4)$ case, we find a rather interesting symmetry in the correlation functions. Interchange between the insertion of $\mathcal{O}_{W_3}$ and $\mathcal{O}_{W_4}$ (in words of Schubert cycle of $Gr(2,4)$,$W_3$ and $W_4$ correspond to $\sigma_{1,1}$ and $\sigma_2$) does not change correlation functions. This is natural because in classical geometry we can not distinguish $\sigma_{1,1}$ from $\sigma_2$ in its algebraic structure. In algebraic geometry, $Gr(2,4)$ describes lines in $CP^3$. $\sigma_2$ correspond to the set of lines passing through a point $p_0$ and $\sigma_{1,1}$ to the set of lines contained fixed plane $h_0$. And the symmeetry of the above two sets comes from duality, i.e we can see $(a_1 : a_2 : a_3 : a_4)$ as a point of $CP^3$ or as a linear form on it. After the completion of this manuscript, a paper by Di Francesco and Itzykson "Quantum intersection rings" has appeared which overlaps with our work.


### Acknowledgement

We would like to thank Prof.T.Eguchi for suggesting this problem and kind encouragement.We also thank Dr.K.Hori for many useful discussions. We are indebted to Dr.T.Hotta and Dr.T.Izubuchi for manipulation of computers.


# A Derivation of Initial Conditions and Some Direct Counting of Amplitudes

We first show $\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle = 1$ for $CP^3$ (resp. $\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle = 1$ for $CP^4$). From (2.9) this is just number of lines passing through two points of $CP^3$(resp. $CP^4$), so it equals to 1 trivially. But we derive this result using schubert calculus of $Gr(2,4)$ (resp.$Gr(2,5)$) which corresponds to the space of lines in $CP^3$ (resp.$CP^4$). Schubert

cycles $\sigma_{a_1,a_2} \subseteq Gr(2,N)$ $(N-2 \geq a_1 \geq a_2 \geq 0)$ form a basis of $H^*(Gr(2,N),Z)$ and are given by following definition.

$$\sigma_{a_1,a_2} = \{l \in Gr(2,N) | dim_C(l \cap V_{N-2+i-a_i}) \geq i\} \tag{A.35}$$

where $V_i$'s are linear subspace of $C^N$ of dimension $i$ satisfying following condition.

$$V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset C^N \tag{A.36}$$

From this definition, subset of $Gr(2,N)$ passing through a point of $CP^{N-1}$ is given as $\sigma_{N-2,0}$ because this condition is equivalent to $dim_C(l \cap V_1) = 1$. Then we can calculate $\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle$ for $CP^3$ (resp.$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle$ for $CP^4$) as follows.

$$\begin{aligned} \langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle &= {}^\sharp(\sigma_{2,0}\cdot\sigma_{2,0})_{Gr(2,4)} = {}^\sharp(\sigma_{2,2})_{Gr(2,4)} = 1 \\ \langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle &= {}^\sharp(\sigma_{3,0}\cdot\sigma_{3,0})_{Gr(2,5)} = {}^\sharp(\sigma_{3,3})_{Gr(2,5)} = 1 \end{aligned} \tag{A.37}$$

In this derivation, we used Pieri's formula

$$\sigma_{a,0}\cdot\sigma_{b_1,b_2} = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ c_1+c_2=a+b_1+b_2}} \sigma_{c_1,c_2} \tag{A.38}$$

and $\sigma_{N-2,N-2}$ corresponds to a point of $Gr(2,N)$.

Next we derive $\langle \mathcal{O}_{W_5}\mathcal{O}_{W_6}\rangle = 1$ for $Gr(2,4)$. Using Plücker map, Gr(2,4) can be embedded in $CP^5$ as a quadratic hypersurface. This embedding is constructed as follows. We map a line $\{\boldsymbol{v}_1,\boldsymbol{v}_2\}_C$ in $CP^3(C^4)$ to a multivector $\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \in \bigwedge^2 C^4$ This map (we call it $\iota$) is injective and conversely the image of a line in $\bigwedge^2 C^4$ is characterized by decomposability, i.e. $\omega \in \bigwedge^2 C^4$ is in $Im(\iota)$ iff $\omega$ can be written as $\omega = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2$. It can be shown that this condition is equiivalent to $\omega \wedge \omega = 0$. So using a basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\}$ of $C^4$ and expanding $\omega$ as follows,

$$\omega = \lambda_{12}\boldsymbol{e}_1\wedge\boldsymbol{e}_2 + \lambda_{13}\boldsymbol{e}_1\wedge\boldsymbol{e}_3 + \lambda_{14}\boldsymbol{e}_1\wedge\boldsymbol{e}_4 + \lambda_{23}\boldsymbol{e}_2\wedge\boldsymbol{e}_3 + \lambda_{24}\boldsymbol{e}_2\wedge\boldsymbol{e}_4 + \lambda_{34}\boldsymbol{e}_3\wedge\boldsymbol{e}_4 \tag{A.39}$$

we can realize $Gr(2,4)(\simeq Im(\iota))$ in $CP^5$ as follows.

$$\begin{aligned} & \omega\wedge\omega = 0 \\ \Longleftrightarrow\ & \lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0 \end{aligned} \tag{A.40}$$

In summary, we can see $Gr(2,4)$ as a quadratic hypersurface $G$ in $CP^5$. Then we want to find the realization of $\sigma_{2,1}(=W_5)$ and $\sigma_{2,2}(=W_6)$ in $G$. From the study of the structure of $G$ (see Chap.6 of Griffith Harris [13]) $\sigma_{2,1}$ corresponds to a line in $G$ and trivially $\sigma_{2,2}$ to a point. If we consider plane $h$ (resp. line $l$) in $CP^5$, quadratic feature of $G$ makes the intersection $(h \cap G)$ (resp.$(l \cap G)$) into conic of $G$ (resp. two points of $G$).Then we have to devide them by factor 2,i.e.

$$\sigma_{2,1} \leftrightarrow \frac{1}{2}(h \cap G) \tag{A.41}$$

$$\sigma_{2,2} \leftrightarrow \frac{1}{2}(l \cap G) \tag{A.42}$$

The space of lines in $G$ (we denote it as $L_G$) is constructed as the subspace of $Gr(2,6)$(space of lines in $CP^5$) using bundle calculation (see [12]).

$$L_G = c_T(Sym^2(U^*)) = 4\tilde{\sigma}_{2,1} \tag{A.43}$$

where $U$ is the universal bundle of $Gr(2,6)$.

(We denote schubert cycles of $Gr(2,6)$ as $\tilde{\sigma}_{a_1,a_2}$ in order to distinguish them from the ones of $Gr(2,4)$).

From (A.42), in $L_G$, to count the number of lines which passes through $\sigma_{2,1}$ and $\sigma_{2,2}$ are equivalent to picking up the lines which passes through $h$ and $l$ (multiplied by factor $\frac{1}{2}$). Then we have

$$\langle \mathcal{O}_{W_5}\mathcal{O}_{W_6}\rangle = {}^\sharp(\frac{1}{2}\tilde{\sigma}_3 \cdot \frac{1}{2}\tilde{\sigma}_2 \cdot 4\tilde{\sigma}_{2,1}) = 1 \tag{A.44}$$

Lastly, using this technique, we calculate the topological amplitude of $d=1$ sector for $CP^3$ and $CP^4$.

$\underline{CP^3}$

$$\begin{aligned}
&\mathcal{O}_{W_3} \leftrightarrow \sigma_1 \quad \mathcal{O}_{W_4} \leftrightarrow \sigma_2 \quad (\text{in}Gr(2,4))\\
&\langle \mathcal{O}_{W_4}\mathcal{O}_{W_4}\rangle = {}^\sharp(\sigma_2 \cdot \sigma_2) = 1\\
&\langle \mathcal{O}_{W_3}^2\mathcal{O}_{W_4}\rangle = {}^\sharp(\sigma_1^2 \cdot \sigma_2) = 1\\
&\langle \mathcal{O}_{W_3}^4 = {}^\sharp(\sigma_1^4) = 2
\end{aligned}$$

$\underline{CP^4}$

$$\begin{aligned}
&\mathcal{O}_{W_3} \leftrightarrow \sigma_1 \quad \mathcal{O}_{W_4} \leftrightarrow \sigma_2 \quad \mathcal{O}_{W_5} \leftrightarrow \sigma_3 \quad (\text{in}Gr(2,5))\\
&\langle \mathcal{O}_{W_5}\mathcal{O}_{W_5}\rangle = {}^\sharp(\sigma_3 \cdot \sigma_3) = 1\\
&\langle \mathcal{O}_{W_3}^3\mathcal{O}_{W_5}\rangle = {}^\sharp(\sigma_1^3 \cdot \sigma_3) = 1\\
&\langle \mathcal{O}_{W_4}^3\rangle = {}^\sharp(\sigma_2^3) = 1\\
&\langle \mathcal{O}_{W_3}\mathcal{O}_{W_4}\mathcal{O}_{W_5}\rangle = {}^\sharp(\sigma_1 \cdot \sigma_2 \cdot \sigma_3) = 1\\
&\langle \mathcal{O}_{W_3}^2\mathcal{O}_{W_4}^2\rangle = {}^\sharp(\sigma_1^2 \cdot \sigma_2^2) = 2\\
&\langle \mathcal{O}_{W_3}^4\mathcal{O}_{W_4}\rangle = {}^\sharp(\sigma_1^4 \cdot \sigma_2) = 3\\
&\langle \mathcal{O}_{W_3}^5\rangle = {}^\sharp(\sigma_1^5) = 5
\end{aligned} \tag{A.45}$$

# References


[1] M.Kontsevich , Y.Manin. *Gromov–Witten Classes,Quantum Cohomology,and Enumerative Geometry* Commun.Math.Phys.164 (1994) 525;

[2] R.Dijkgraaf, E.Witten. Nuclear Physics B342 (1990) 486;

[3] C.Itzykson. *Counting rational curves on rational surfaces* Saclay preprint T94/001;

[4] S.Cecotti,C.Vafa. Nuclear Physics B367 (1991) 359;

[5] B.Dubrovin. *Geometry of 2D Topological Field Theory* Preprint SISSA-89/94/FM hepth 9407018;

[6] C.Vafa. *Topological Mirrors and Quantum Rings* HUTP-91/A059;

[7] E.Witten. *Topological Sigma Models* Commun.Math.Phys.118(1988)411;

[8] E.Witten. *On the Structure of the Topological Phase of Two Dimensional Gravity* Nucl.Phys.B342(1990)486;

[9] T.Eguchi,S.K.Yang. $N = 2$ *Super Conformal Models as Topological Field Theories* Mod.Phys.Lett.A Vol.5 No.21(1990)1693;

[10] W.Lerche,C.Vafa,N.Warner. *Chiral Rings in* $N = 2$ *Super Conformal Field Theories* Nucl.Phys.B324(1989)427;

[11] K.Hori. *Constraints For Topological Strings In* $D \geq 1$ UT-694;

[12] M.Jinzenji,M.Nagura. *Mirror Symmetry and An Exact Calculation of* $N-2$ *Point Correlation Function on Calabi-Yau Manifold Embedded in* $CP^{N-1}$ UT-6**;

[13] P.Griffith,J.Harris. *Principles of Algebraic Geometry* J.Wiley,N.Y.,1978;